HIGH SCHOOL ALGEBRA: AUSSIE FIR TREE

UNIT OVERVIEW

The Aussie Fir Tree task is a culminating task for a 2-3 week unit on algebra that uses the investigation of growing patterns as a vehicle to teach students to visualize, identify and describe real world mathematical relationships. Students who demonstrate mastery of the unit are able to solve the Aussie Fir Tree task in one class period.

TASK DETAILS

Task Name: Aussie Fir Tree

Grade: High School

Subject: Algebra

Task Description: This task asks students to recognize geometric patterns, visualize and extend the pattern, generate a non-linear sequence, develop and algebraic generalization that models the growth of a quadratic function and verify the inverse relationship of the quadratic relationship.

Standards Assessed:
F.BF.1 Write a function that describes a relationship between two quantities.
F.BF.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.
F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.
A.CED.1 Create equations in one variable and use them to solve problems. Include equations arising from linear and quadratic functions.
A.REI.4 Solve quadratic equations in one variable.

Standards for Mathematical Practice:
MP.1 Make sense of problems and persevere in solving them.
MP.3 Construct viable arguments and critique the reasoning of others.
MP.4 Model with mathematics.
MP.6 Attend to precision.
MP.7 Look for and make use of structure.

Department of Education
Dennis M. Walcott, Chancellor

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The task and instructional supports in the following pages are designed to help educators understand and implement tasks that are embedded in Common Core-aligned curricula. While the focus for the 2011-2012 Instructional Expectations is on engaging students in Common Core-aligned culminating tasks, it is imperative that the tasks are embedded in units of study that are also aligned to the new standards. Rather than asking teachers to introduce a task into the semester without context, this work is intended to encourage analysis of student and teacher work to understand what alignment looks like. We have learned through the 2010-2011 Common Core pilots that beginning with rigorous assessments drives significant shifts in curriculum and pedagogy. Universal Design for Learning (UDL) support is included to ensure multiple entry points for all learners, including students with disabilities and English language learners.

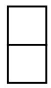
Acknowledgements: The unit outline was developed by George Georgilakis (CFN 206) and Michele Luard (CFN 110) with input from Curriculum Designers Alignment Review Team. The tasks were developed by the schools in the 2010-2011 NYC DOE High School Performance Based Assessment Pilot, in collaboration with the Silicon Valley Mathematics Initiative and SCALE.

HIGH SCHOOL ALGEBRA: AUSSIE FIR TREE PERFORMANCE TASK

The Aussie Fir Tree
It grows down under

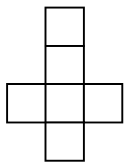
Consider the following function that generates the geometric pattern of a reverse growing fir tree.

Stage 1



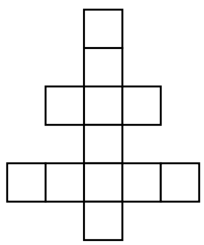
2 unit squares

Stage 2



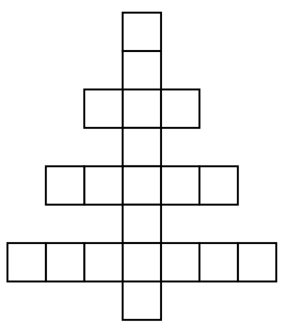
6 unit squares

Stage 3



12 unit squares

Stage 4



1. Draw and describe **Stage 5** of the pattern in terms of its shape and number of unit squares needed to construct the fir tree.

The Aussie Fir Tree

It grows down under

2. Describe how the pattern is growing?

3. How many unit squares are needed to build a **Stage 10** Aussie Fir Tree? Show your work.

4. Given any stage number **n**, determine a closed form equation to determine the amount of unit squares needed to build the tree.

5. Your mate tells you that exactly 274 unit squares will make an Aussie Fir Tree. He is wrong. Explain to him why his statement is false.

HIGH SCHOOL ALGEBRA: AUSSIE FIR TREE

UNIVERSAL DESIGN FOR LEARNING (UDL) PRINCIPLES

Aussie Fir Tree – Math Grade 12
Common Core Learning Standards/
Universal Design for Learning

The goal of using Common Core Learning Standards (CCLS) is to provide the highest academic standards to all of our students. Universal Design for Learning (UDL) is a set of principles that provides teachers with a structure to develop their instruction to meet the needs of a diversity of learners. UDL is a research-based framework that suggests each student learns in a unique manner. A one-size-fits-all approach is not effective to meet the diverse range of learners in our schools. By creating options for how instruction is presented, how students express their ideas, and how teachers can engage students in their learning, instruction can be customized and adjusted to meet individual student needs. In this manner, we can support our students to succeed in the CCLS.

Below are some ideas of how this Common Core Task is aligned with the three principles of UDL; providing options in representation, action/expression, and engagement. As UDL calls for multiple options, the possible list is endless. Please use this as a starting point. Think about your own group of students and assess whether these are options you can use.

REPRESENTATION: *The "what" of learning.* How does the task present information and content in different ways? How students gather facts and categorize what they see, hear, and read. How are they identifying letters, words, or an author's style?

In this task, teachers can…

- ✓ **Provide physical objects for and spatial models to convey perspective or interaction** by providing students with concrete and/or virtual square tiles.

ACTION/EXPRESSION: *The "how" of learning.* How does the task differentiate the ways that students can express what they know? How do they plan and perform tasks? How do students organize and express their ideas?

In this task, teachers can…

- ✓ **Provide calculators, graphing calculators, geometric sketchpads, or pre-formatted graph paper** to provide students with multiple tools to identify linear and quadratic relationships in a realistic context.

ENGAGEMENT: *The "why" of learning.* How does the task stimulate interest and motivation for learning? How do students get engaged? How are they challenged, excited, or interested?

In this task, teachers can…

- ✓ **Provide feedback that models how to incorporate evaluation, including identifying patterns of errors and wrong answers, into positive strategies for future success** by encouraging peers to share their strategies to improve individual solutions.

Visit <u>http://schools.nyc.gov/Academics/CommonCoreLibrary/default.htm</u> to learn more information about UDL.

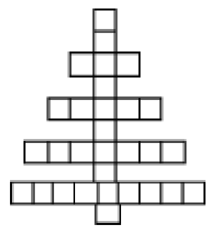
HIGH SCHOOL ALGEBRA: AUSSIE FIR TREE RUBRIC

The rubric section contains a scoring guide and performance level descriptions for the Aussie Fir Tree task.

Scoring Guide: The scoring guide is designed specifically to each small performance task. The points highlight each specific piece of student thinking and explanation required of the task and help teachers see common misconceptions (which errors or incorrect explanations) keep happening across several papers. The scoring guide can then be used to refer back to the performance level descriptions.

Performance Level Descriptions: Performance level descriptions help teachers think about the overall qualities of work for each task by providing information about the expected level of performance for students. Performance level descriptions provide score ranges for each level, which are assessed using the scoring guide.

Aussie Fir Tree Scoring Guide

The Aussie Fir Tree		Rubric
The elements of performance required by this task are: • Recognizes geometric patterns. · Visualizes, extends and describes patterns. · Determines a solution to a polynomial relationship. · Develops an algebraic equation that models the growth of a quadratic function. · Verifies the inverse relationship of the polynomial equation.	Points	Section Points
1. Draws stage 5 with 30 unit squares.  Describes stage 5 such as: The trunk is ten unit squares tall and there are four sets of branches 3, 5, 7 and 9 units.	1 1	2
2. Describe the growing pattern such as: It grows by consecutive even numbers from stage to stage. The pattern grows between each stage by 2, 4, 6, 8,…	2	2
3. **110** unit squares. Show work such as: Stage 5 is 30 or 5 • 6, Stage 6 is 42 or 6 • 7, Stage 7 is 56, or 7•8 so, Stage 10 is 10 • 11 = 110 unit squares.	1 1	2
4. **Unit squares = n(n+1)** *Partial Credit* Provide a correct expression	2 (1)	2
5. Provides an explanation such as: No two consecutive integers when multiplied together equals 274. The consecutive integers 16 • 17 = 272. The next smallest product would be 17 • 18 = 306, which is too large.	2	2
Total Points		**10**

High School Algebra: Aussie Fir Tree
Rubric

Performance Level Descriptions and Cut Scores

Performance is reported at four levels: 1 through 4, with 4 as the highest.

Level 1: Demonstrates Minimal Success (0-3 points)

The student's response shows few of the elements of performance that the tasks demand. The work shows a minimal attempt on the problem and struggles to make a coherent attack on the problem. Communication is limited and shows minimal reasoning. The student's response rarely uses definitions in their explanations. The student struggles to recognize patterns or the structure of the problem situation.

Level 2: Performance below Standard (4-5 points)

The student's response shows some of the elements of performance that the tasks demand and some signs of a coherent attack on the core of some of the problems. However, the shortcomings are substantial and the evidence suggests that the student would not be able to produce high-quality solutions without significant further instruction. The student might ignore or fail to address some of the constraints. The student may occasionally make sense of quantities in relationships in the problem, but their use of quantity is limited or not fully developed. The student response may not state assumptions, definitions, and previously established results. While the student makes an attack on the problem, it is incomplete. The student may recognize some patterns or structures, but has trouble generalizing or using them to solve the problem.

Level 3: Performance at Standard (6-7 points)

For most of the task, the student's response shows the main elements of performance that the tasks demand and is organized as a coherent attack on the core of the problem. There are errors or omissions, some of which may be important, but of a kind that the student could well fix, with more time for checking and revision and some limited help. The student explains the problem and identifies constraints. The student makes sense of quantities and their relationships in the problem situations. The student often uses abstractions to represent a problem symbolically or with other mathematical representations. The student response may use assumptions, definitions, and previously established results in constructing arguments. They may make conjectures and build a logical progression of statements to explore the truth of their conjectures. The student might discern patterns or structures and make connections between representations.

Level 4: Achieves Standards at a High Level (8-10 points)

The student's response meets the demands of nearly the entire task, with few errors. With some more time for checking and revision, excellent solutions would seem likely. The student response shows understanding and use of stated assumptions, definitions and previously established results in construction arguments. The student is able to make conjectures and build a logical progression of statements to explore the truth of their conjecture. The student response routinely interprets their mathematical results in the context of the situation and reflects on whether the results make sense. The communication is precise, using definitions clearly. Students look closely to discern a pattern or structure. The body of work looks at the overall situation of the problem and process, while attending to the details.

HIGH SCHOOL ALGEBRA: AUSSIE FIR TREE
ANNOTATED STUDENT WORK

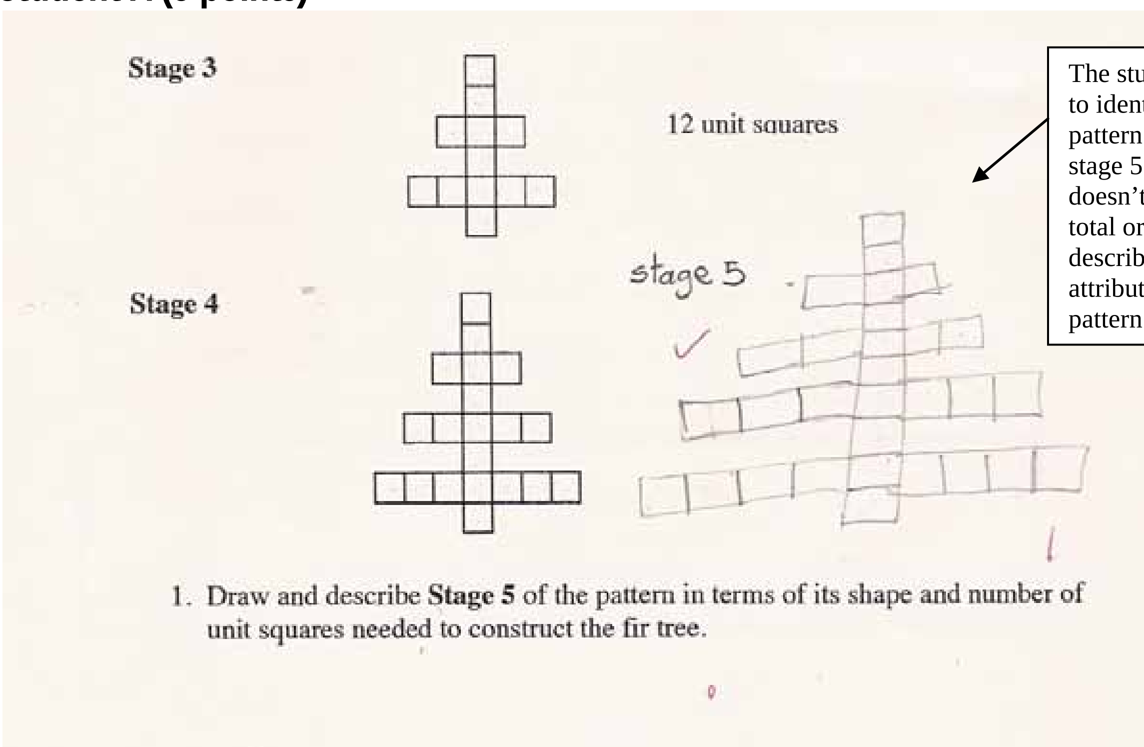
This section contains annotated student work at a range of score points and implications for instruction for each performance level. The student work shows examples of student understandings and misunderstandings of the task, which can be used with the implications for instruction to understand how to move students to the next performance level.

High School Algebra: Aussie Fir Tree
Annotated Student Work

Level 4: Achieves Standards at a High Level (Score Range 8 – 10)

The student's response meets the demands of nearly the entire task, with few errors. With some more time for checking and revision, excellent solutions would seem likely. The student response shows understanding and use of stated assumptions, definitions and previously established results in construction arguments. The student is able to make conjectures and build a logical progression of statements to explore the truth of their conjecture. The student response routinely interprets their mathematical results in the context of the situation and reflects on whether the results make sense. The communication is precise, using definitions clearly. The students looks closely to discern a pattern or structure. The body of work looks at the overall situation of the problem and process, while attending to the details.

Student A (9 points)



The student is able to identify the visual pattern and draw stage 5. The student doesn't count the total or attempt to describe the attributes of how the pattern grows.

High School Algebra: Aussie Fir Tree
Annotated Student Work

2. Describe how the pattern is growing?

from the stage 1 to stage 2 "4" is added, from stage 2 to stage 3 "6" is added, from stage 3 to stage 4 "8" is added.

2

In part 2 the student notices that the pattern grows by consecutive even numbers.

3. How many unit squares are needed to build a **Stage 10** Aussie Fir Tree? Show your work.

Stage	unit squares
5	30
6	42
7	56
8	72
9	90
10	110

we need 110 unit squares ~~to~~ build stage 10. ~~because~~ from stage 1 to 2 you added a 4 from stage 2 to 3 you added a 6 and so on. you keep adding a 2 to the number you added.

The student uses this pattern of increasing consecutive even numbers to find the total number of squares to build stage 10.

4. Given any stage number **n**, determine a closed form equation to determine the amount of unit squares needed to build the tree.

$\boxed{y = (n+1) \cdot n}$

1

The student uses correct algebraic notation to write a closed equation in part 4. (CCSS Creating Equations)

5. Your mate tells you that exactly 274 unit squares will make an Aussie Fir Tree. He is wrong. Explain to him why his statement is false.

Performance Task The Aussie Fir Tree P 2

His statement is false because none of the stage number can have 274 unit. The stage number 16 has 272 unit but that less than 274 unit. And stage number 17 has 288 unit and that's more than 274 unit.

In part 5 the student uses the equation and substitution to show why 274 is not a possible solution. A better argument would have also included the pattern above as well the pattern below to disprove the statement. (MP 3)

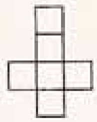
High School Algebra: Aussie Fir Tree
Annotated Student Work

Student B (8 points)

Stage 1 x2

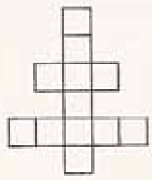
2 unit squares

Stage 2 x3



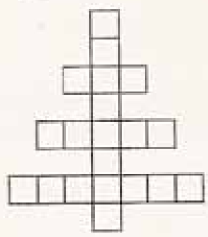
6 unit squares

Stage 3 x4



12 unit squares

Stage 4 x5



20 units

1. Draw and describe **Stage 5** of the pattern in terms of its shape and number of unit squares needed to construct the fir tree.

0 1

Stage 5 has one more branch with an extra square on each side.

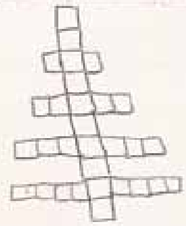
1

Stage 5 = 5x6 = 30 units. ✓

> The student uses the information given on page one to look at patterns and to experiment with numerical expressions to get the given totals. The student then uses the pattern to generate the total squares without counting. The student is able to identify the visual pattern.

High School Algebra: Aussie Fir Tree
Annotated Student Work

Stage 5
5x6



30 units.

✓

In part 2 the student jumps right to the generalization.

2. Describe how the pattern is growing?

Each branch has one more unit on each side and one extra on the bottom.

2

The stage # multiplied by stage # + 1 gives the amount of unit squares needed

3. How many unit squares are needed to build a **Stage 10** Aussie Fir Tree? Show your work.

$n(n+1)$

Stage 10 → $10 \times 11 = 110$ units

1

1

The student uses the generalization (from part 2) to calculate the total for stage 10.

4. Given any stage number **n**, determine a closed form equation to determine the amount of unit squares needed to build the tree.

$n(n+1) = a$

2

a = amount of unit squares needed.

The student uses variables and correct notation to write a closed equation in part 4. (CCSS Creating Equations) Notice that the student defines the variable not given in the stem of the prompt.

5. Your mate tells you that exactly 274 unit squares will make an Aussie Fir Tree. He is wrong. Explain to him why his statement is false.

Because no 2 consecutive #s multiply to equal 274.

2 0

Performance Task The Aussie Fir Tree P 2

8

The student could give more detail in part 5, making a convincing argument. (MP 3)

Level 4 Implications for Instruction

Students need more opportunities to develop justification and make convincing arguments. They should have frequent opportunities to question, to critique, and to improve the arguments of others. "*Does this convince you? Why or Why not? What would make it more convincing? Why do you disagree? How could you convince the person to change their mind?*"

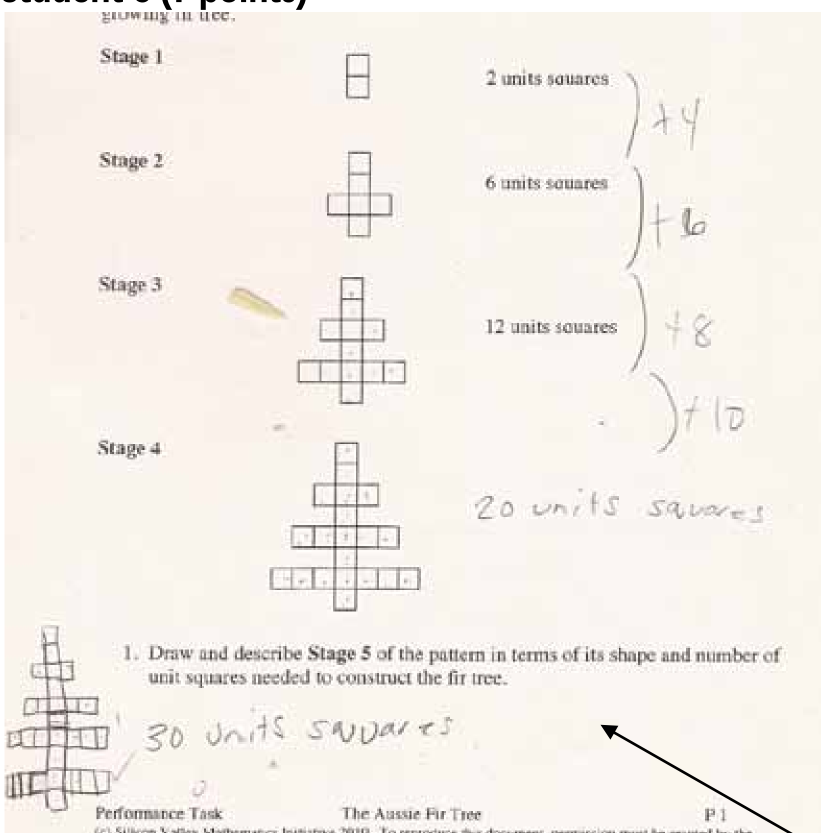
Students at this level should be encouraged to give algebraic as well as numeric ideas to make their justifications.

High School Algebra: Aussie Fir Tree
Annotated Student Work

Level 3: Performance at Standard (Score Range 6 – 7)

For most of the task, the student's response shows the main elements of performance that the tasks demand and is organized as a coherent attack on the core of the problem. There are errors or omissions, some of which may be important, but of a kind that the student could well fix, with more time for checking and revision and some limited help. The student explains the problem and identifies constraints. The student makes sense of quantities and their relationships in the problem situations. They often use abstractions to represent a problem symbolically or with other mathematical representations. The student response may use assumptions, definitions, and previously established results in constructing arguments. They may make conjectures and build a logical progression of statements to explore the truth of their conjectures. The student might discern patterns or structures and make connections between representations.

Student C (7 points)


Stage 1

2 units squares

+4

Stage 2

6 units squares

+6

Stage 3

12 units squares

+8

+10

Stage 4

20 units squares

1. Draw and describe **Stage 5** of the pattern in terms of its shape and number of unit squares needed to construct the fir tree.

30 units squares

Performance Task The Aussie Fir Tree P.1

The student is able to identify the visual pattern and count the squares in part 1, but does not attempt to describe how the attributes of the shape grow.

High School Algebra: Aussie Fir Tree
Annotated Student Work

In part 2 the student notes clearly the pattern of increasing consecutive even numbers and can use this pattern to make a convincing argument in part 5.

However the student can't move past the recursive pattern to find an equation in part 4. (CCSS Equations and Inequalities) The student seems to attempt to quantify an algebraic pattern for stage 3, which does not correlate to work in other parts of the paper. The student doesn't seem to connect the idea of equation as a tool for generalizing to all cases.

The student is also able to use the pattern of increasing consecutive even numbers to find the total for stage 10.

The Aussie Fir Tree
It grows down under

2. Describe how the pattern is growing?

$2 \xrightarrow{+4} 6 \xrightarrow{+6} 12 \xrightarrow{+8} 20$

The Pattern is quadratatic, and it is increasing by 2 in the second diffrency

2

3. How many unit squares are needed to build a **Stage 10** Aussie Fir Tree? Show your work.

$2 \xrightarrow{+4} 6 \xrightarrow{+6} 12 \xrightarrow{+8} 20 \xrightarrow{+10} 30 \xrightarrow{+12} 42 \xrightarrow{+14} 56 \xrightarrow{+16} 72 \xrightarrow{+18} 90 \xrightarrow{+20} 110$

Stage 10 = 110

2

4. Given any stage number n, determine a closed form equation to determine the amount of unit squares needed to build the tree.

Stage # = 3 n+3 = X

I put stage 3 for an example and added it by n so the answer will be x until I find what n equals.

0

5. Your mate tells you that exactly 274 unit squares will make an Aussie Fir Tree. He is wrong. Explain to him why his statement is false.

$110 \xrightarrow{+22} 132 \xrightarrow{+24} 156 \xrightarrow{+26} 182 \xrightarrow{+28} 210 \xrightarrow{+30} 240 \xrightarrow{+32} 272$

You can't make one because it would be 2 units short.

2

Performance Task The Aussie Fir Tree P 2

7

High School Algebra: Aussie Fir Tree
Annotated Student Work

Level 3 Implications for Instruction

Students at this level still struggle with giving good verbal descriptions of what they see. The biggest difficulties were in finding generalizable rules rather than recursive rules. One helpful way to do this is to ask students break down the pattern into simpler parts. "*What do you see when you look at the pattern? How can you decompose the shape into parts?. . . . How does the tree trunk grow? How could we use algebra to describe how to find the tree trunk for any pattern number?"* Breaking down the problem into simpler steps helps students manage the thinking in smaller chunks.

Students need opportunities to compare and contrast strategies and to evaluate their usefulness. So during class discussions it is important not to stop after several students have shared how they solved the problems. Teachers need to ask students to think about and reflect on the strategies. "*Which one is easier to use? Why? Which one would be most useful for finding the 1000th term? Why? What is the same about both formulas or strategies? What is different? How can we be sure that they will always give the same solution?"*

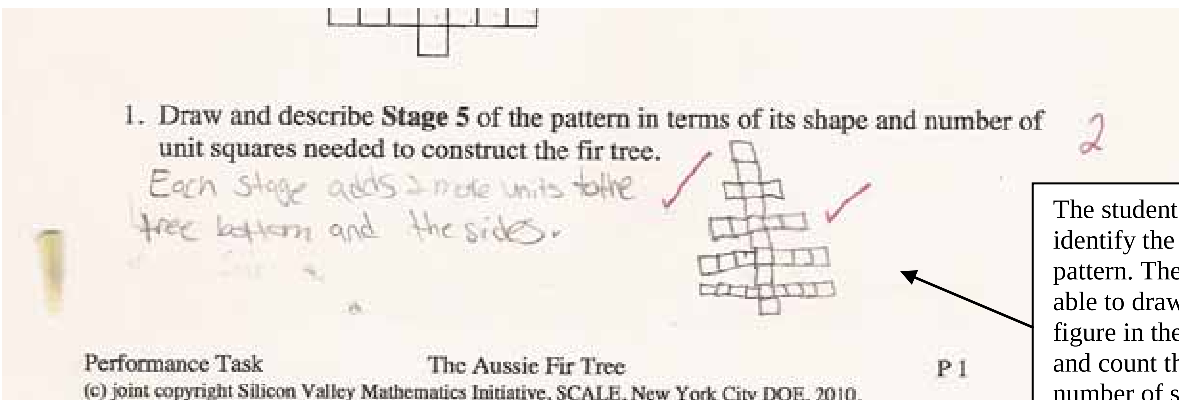
Students need more opportunities to develop justification and make convincing arguments. They should have frequent opportunities to question, to critique, and to improve the arguments of others. "*Does this convince you? Why or Why not? What would make it more convincing? Why do you disagree? How could you convince the person to change their mind?"*

High School Algebra: Aussie Fir Tree
Annotated Student Work

Level 2: Performance below Standard (Score Range 4-5)

The student's response shows some of the elements of performance that the tasks demand and some signs of a coherent attack on the core of some of the problems. However, the shortcomings are substantial, and the evidence suggests that the student would not be able to produce high-quality solutions without significant further instruction. The student might ignore or fail to address some of the constraints. The student may occasionally make sense of quantities in relationships in the problem, but their use of quantity is limited or not fully developed. The student response may not state assumptions, definitions, and previously established results. While the student makes an attack on the problem it is incomplete. The student may recognize some patterns or structures, but has trouble generalizing or using them to solve the problem.

Student D (4 points)



1. Draw and describe **Stage 5** of the pattern in terms of its shape and number of unit squares needed to construct the fir tree. 2

Each stage adds 2 more units to the tree bottom and the sides.

Performance Task The Aussie Fir Tree P 1

(c) joint copyright Silicon Valley Mathematics Initiative, SCALE, New York City DOE, 2010.

The student is able to identify the visual pattern. The student is able to draw the next figure in the pattern and count the total number of squares in the drawing. The student notices that the middle row increases by 2 each time and that each new branch adds another square on each side.

High School Algebra: Aussie Fir Tree
Annotated Student Work

The student isn't quite able to come up with the idea of adding consecutive even numbers each time, but gives a minimal description tied to the structure of the visual pattern.

2. Describe how the pattern is growing? ✓ 2

The tree grows with 2 units added to the bottom at every stage. However many units were previously at the side there's also 2 more units added to that number to put at the next stage.

3. How many unit squares are needed to build a **Stage 10** Aussie Fir Tree? Show your work. x 0

Stage 9 – 78 units

To build a stage 10 Aussie Fir Tree you need 98 unit squares.

78
+20
98

stage 10 – 98

The student also shows this understanding of consecutive even numbers by choosing to add 20 to stage 9 to get stage 10. Unfortunately the student has not shown enough work, but appears to have made a calculation error somewhere between stage 5 and stage 9.

The student does not attempt to write an equation. In part 3 he uses a recursive idea of adding 2, which matches to either increasing the side of the branch, but does not include the idea that new branches are added, or thinks about the vertical branch increasing by 2. (CCSS Seeing Structure in Expressions) The student needs substantial instruction to think about writing an equation for the entire situation.

4. Given any stage number **n**, determine a closed form equation to determine the amount of unit squares needed to build the tree. x 0

5. Your mate tells you that exactly 274 unit squares will make an Aussie Fir Tree. He is wrong. Explain to him why his statement is false. x 0

Performance Task The Aussie Fir Tree P 2

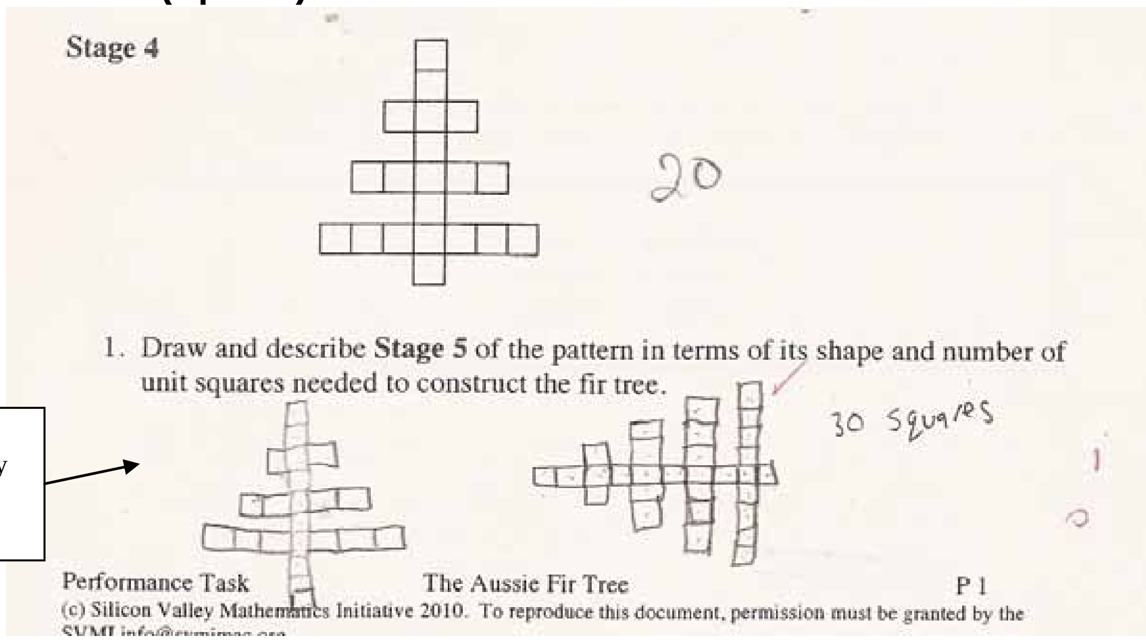
He is wrong because he needs to know what stage and the number of units from the previous stage to know how much unit squares makes up one tree/stage.

4

The recursive thinking is continued in part 5. (CCSS Creating Equations)

High School Algebra: Aussie Fir Tree
Annotated Student Work

Student E (5 points)



Stage 4

20

1. Draw and describe **Stage 5** of the pattern in terms of its shape and number of unit squares needed to construct the fir tree.

30 squares

1

0

Performance Task The Aussie Fir Tree P 1

The student is able to identify the visual pattern.

High School Algebra: Aussie Fir Tree
Annotated Student Work

The student is much more descriptive in the size of the sides, stage number minus one. This quantification sets a foundation for writing the equation.

2. Describe how the pattern is growing?

It is adding two more squares down but on the top of the two it is adding the amount of square equal to the stage number minus one on each side.

2

3. How many unit squares are needed to build a **Stage 10** Aussie Fir Tree? Show your work. It would need 110 squares.

x	y	
1	2	+4
2	6	+6
3	12	+8
4	20	+10
5	30	

x	y	
5	30	+12
6	42	+14
7	56	+16
8	72	+18
9	90	+20
10	110	

The student chooses to use a table to extend the pattern of adding consecutive even numbers to total squares for stage 10.

4. Given any stage number **n**, determine a closed form equation to determine the amount of unit squares needed to build the tree.

x = stage number
y = total # of squares

$x^2 + (x = y$

0 2

The student can then use the information from the table to write an equation for the pattern. (MP5)

5. Your mate tells you that exactly 274 unit squares will make an Aussie Fir Tree. He is wrong. Explain to him why his statement is false.

His statement is wrong because if you use the equation above then it wouldn't be able to do the equation backwards the closest to 274 units would be 260. That would mean it wouldn't make a Aussie fir tree. This is proven is the table on the left.

X 0

11 × 11 = 121

x	y	
10	110	+22
11	132	+24
12	164	+26
13	190	+26
14	216	+28
15	248	+30
16	260	+32

The student uses addition to prove why the statement is incorrect in part 5, which could have led to the convincing argument without the small arithmetic error. (MP 3) The student didn't check the rule using the equation generated in part 4. (MP 1)

Performance Task The Aussie Fir Tree P 2

5

Level 2 Implications for Instruction

Many students at this level are using recursive rules to extend the pattern. While this is helpful and easy to use for a small number of cases, it is cumbersome and prone to errors when trying to extend the pattern for larger numbers. Students need to see that it is more helpful to find more generalizable rules to solve problems for all cases.

One helpful way to do this is to ask students break down the pattern into simpler parts. "*What do you see when you look at the pattern? How can you decompose the shape into parts? How does the tree trunk grow? How could we use algebra to describe how to find the tree trunk for any pattern number?"* Breaking down the problem into simpler steps helps students manage the thinking in smaller chunks. But students in this stage need to learn questions to help them progress in their thinking, to develop strategies for finding a generalizable rule.

Students need to be encouraged to give more detail about what they see. So in class the teacher might ask, "*That's interesting, can you tell me a bit more?* **Or** *where do you see the n in the diagram or the (n+1)?"* The more detailed their descriptions usually the easier it is to quantify the ideas symbolically. Students should also be more descriptive in thinking about classes of numbers. In elementary school it is good to notice that numbers are odd or even, but by this grade level students should start to classify numbers as consecutive or consecutive odd numbers, multiples of . . ., powers of . . . , triangular numbers, etc. The types of patterns that students think about should be expanded.

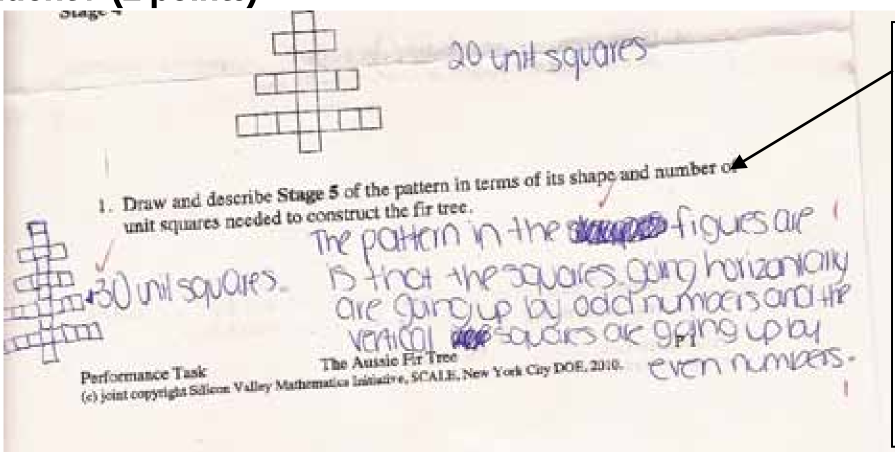
Students need to have experiences thinking about types of linear patterns; those that are proportional and those that are not proportional. Take the work of student 692. The strategy of doubling from case 5 to case 10 works for proportional patterns, but not for patterns with a constant. Students can benefit by looking at two cases at the same time and comparing which one will work by doubling and which one won't. Drawing graphs of the situations can help to clarify this idea.

High School Algebra: Aussie Fir Tree
Annotated Student Work

Level 1: Demonstrates Minimal Success (Score Range 0 –3)

The student's response shows few of the elements of performance that the tasks demand. The work shows a minimal attempt on the problem and struggles to make a coherent attack on the problem. Communication is limited and shows minimal reasoning. The student's response rarely uses definitions in their explanations. The student struggles to recognize patterns or the structure of the problem situation.

Student F (2 points)



Stage 4

20 unit squares

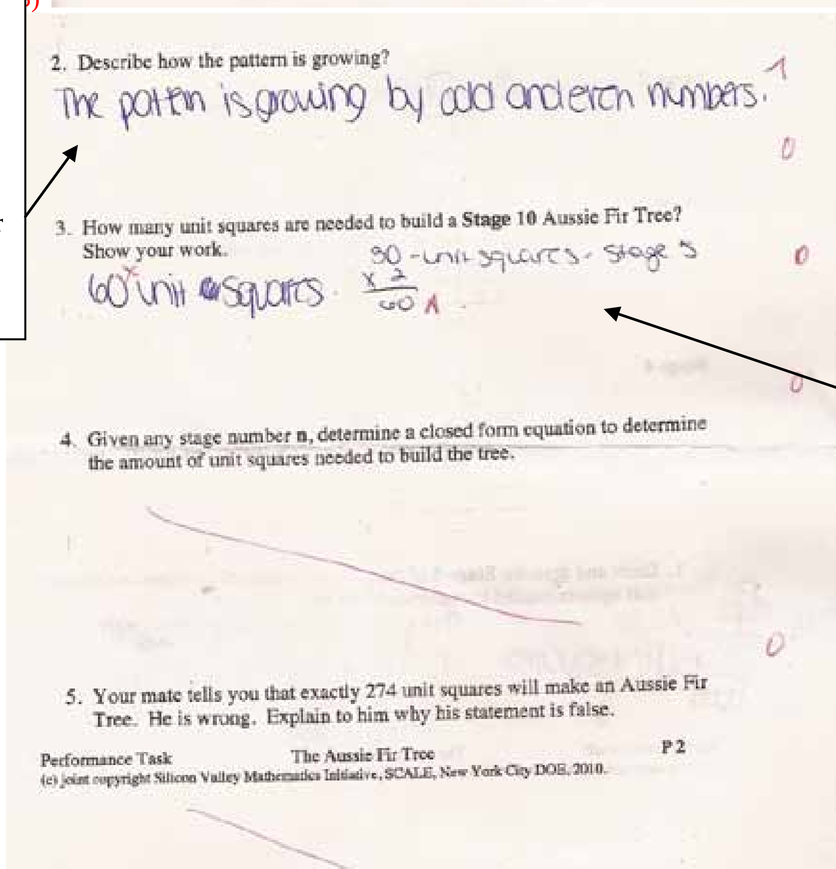
1. Draw and describe **Stage 5** of the pattern in terms of its shape and number of unit squares needed to construct the fir tree.

30 unit squares.

The pattern in the figures are is that the squares going horizontally are going up by odd numbers and the vertical squares are going up by even numbers.

Performance Task The Aussie Fir Tree P1
(c) joint copyright Silicon Valley Mathematics Initiative, SCALE, New York City DOE, 2010.

The student is able to identify the visual pattern. The student is able to draw and count the squares in the next pattern of the sequence. The student notices that each new branch is going up by odd numbers, but doesn't quantify how much to increase each time.



2. Describe how the pattern is growing?

The pattern is growing by odd and even numbers. 1

0

3. How many unit squares are needed to build a **Stage 10** Aussie Fir Tree? Show your work.

60 unit squares

30 - unit squares - stage 5
x 2
60

0

0

4. Given any stage number **n**, determine a closed form equation to determine the amount of unit squares needed to build the tree.

0

5. Your mate tells you that exactly 274 unit squares will make an Aussie Fir Tree. He is wrong. Explain to him why his statement is false.

Performance Task The Aussie Fir Tree P 2
(c) joint copyright Silicon Valley Mathematics Initiative, SCALE, New York City DOE, 2010.

The student notices that the vertical squares go up by an even number, rather than increase by 2's. By grade 9, students should be digging deeper than odd or even for patterns. (MP 1)

The student uses proportional reasoning to increase the pattern from stage 5 to stage 10. The lack of quantitative reasoning at earlier parts of the task doesn't give the student enough detail to see that the pattern is not proportional or even linear. (MP 7 CCSS Seeing Structure in Expressions) The student does not have the perseverance to attempt further parts of the task.

High School Algebra: Aussie Fir Tree
Annotated Student Work

Level 1 Implications for Instruction

Students need help learning to describe accurately and precisely what they see as the visual pattern grows. These descriptions become tools that can be used to describe the pattern algebraically.

Students need ways of organizing their thinking, such as making tables to see how the pattern grows numerically. The work done by student 209 can help to lead to a full solution algebraically, but students need discussions about the purpose of the tools. Students at this level don't know or understand the purpose of the tools, so even when they make a table that aren't sure of what they learn from it that can help them make the generalization. A useful teaching devise is self talk. The teacher starts to solve a problem or a student describing a solution goes part way, then the teacher stops and asks, "What do you think comes next?" Students need to start anticipating how the solution progresses. In this case, students might notice that each stage increases by a consecutive even number.

Students need to talk about cases. When making a conjecture, they need to understand that just one example is not sufficient. They need to test their ideas against several cases to see if it at least holds true for all the examples that they have available. At later stages of their development, they will learn that there are never enough examples to prove a case and that the proof lies in the physical geometry of the pattern. However, many students at this level made a generalization, when they had other examples on their paper that disproved their assertions. Students need frequent opportunities to work with patterns, organize the information for them, and make and test conjectures.

HIGH SCHOOL ALGEBRA: AUSSIE FIR TREE
INSTRUCTIONAL SUPPORTS

The instructional supports on the following pages include a unit outline with formative assessments and suggested learning activities. Teachers may use this unit outline as it is described, integrate parts of it into a currently existing curriculum unit, or use it as a model or checklist for a currently existing unit on a different topic.

Unit Outline –High School Math

INTRODUCTION: This unit outline provides an example of how teachers may integrate performance tasks into a unit. *Teachers may (a) use this unit outline as it is described below; (b) integrate parts of it into a currently existing curriculum unit; or (c) use it as a model or checklist for a currently existing unit on a different topic.*

High School Algebra: Aussie Fir Tree Unit

UNIT TOPIC AND LENGTH:

- This unit uses the investigation of growing patterns as a vehicle to teach students how to visualize, identify, and describe real world mathematical relationships. Students will demonstrate mastery of the content by making sense of the Aussie Fir Tree Performance Task and persevering in solving this task. Suggested unit length 2 – 3 weeks.

COMMON CORE LEARNING STANDARDS:

- F.BF.1 Write a function that describes a relationship between two quantities.★
- F.BF.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.
- F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.
- A.CED.1 Create equations in one variable and use them to solve problem. Include equations arising from linear and quadratic functions.
- A.REI.4. Solve quadratic equations in one variable.
- MP.1 Make sense of problems and persevere in solving them.
- MP. 3 Construct viable arguments and critique the reasoning of others.
- MP. 4 Model* with mathematics.
- MP. 6 Attend to precision.
- MP. 7 Look for and make use of structure.

BIG IDEAS/ENDURING UNDERSTANDINGS:

- Mathematicians use functions and equations to determine and explain patterns of change.
- A pattern of change can be described through a function.
- Equations predict patterns.

ESSENTIAL QUESTIONS:

- How can finding patterns be helpful to me?
- To what extent does my knowledge of functions help me describe patterns of change?
- How can I use equations to predict patterns of change?

Unit Outline –High School Math

CONTENT:

Growth Patterns

- Geometric figure
- Auxiliary lines for solving problems
- Mathematician's application of patterns as problem solving tools

Functions

- recursive
- domains
- Geometric Sequences
- Explicit Expressions

Equations

- Linear Equations
- Quadratic functions
- Simple rational and exponential functions
- Arithmetic Series
- Recursive Process

SKILLS:

- **Visualize** and **create** growing patterns.
- **Describe** and **extend** a pattern.
- **Show and Explain** method of using patterns to find possible solutions to problems.

- **Recognize** that sequences are functions. **Define** a sequence whose domain is a subset of the integers.
- **Write** a function that **describes** a relationship between two quantities.
- **Determine** and **show** an explicit expression.
- **Develop** a recursive process or steps for calculation from a context.

- **Solve** quadratic equations.
- **Create** equations and inequalities in one variable.
- **Solve** equations and inequalities in one variable.
- **Solve** quadratic equations in one variable.
- **Engage** in collaborative discussions with peers.
- **Explain** each step in solving a simple equation as following from the equality of numbers asserted at the previous step.
- **Construct** a viable argument to **justify** a solution method.
- **Use** content vocabulary in explanations.
- **Organize** work using tables and charts.

KEY TERMS / VOCABULARY:

- patterns
- functions
- quadratic equations
- linear equations
- expressions

ASSESSMENT EVIDENCE AND ACTIVITIES:

INITIAL ASSESSMENT: SQUARE PATTERNS

The **initial assessment** also allows for what is sometimes called a *touchstone task*. The task should be rich enough that it can be solved from a variety of approaches so that students can make sense of it in natural ways. Then as the unit progresses, students should be able to move to more efficient or grade-level appropriate strategies. As the students learn new ideas or procedures, students and the teacher can reflect upon how these new ideas and procedures might apply to the initial task. *See the task Square Patterns for task details.*

FORMATIVE ASSESSMENT LESSON: TABLE TILES

Use this Formative Assessment Lesson ¾ of the way through the unit to surface misconceptions and, through the course of the lesson, to provide ways for students to resolve these misconceptions and deepen their understanding. By surfacing misunderstandings the teacher is able to make mid-unit corrections to instruction. Thus, students' experiences help to improve learning rather than waiting until the final assessment to uncover problems or gaps in learning. *See the Formative Assessment Lesson Table Tiles for full details on how to use the lesson.*

FINAL PERFORMANCE TASK: AUSSIE FIR TREE

At the end of the unit, students will be given The Aussie Fir Tree Performance Task to determine how they have improved their thinking and mathematical skills over the course of the instructional unit. This task assesses students' skills in and knowledge of recognizing geometric patterns, visualizing and extending the pattern, generating a non-linear sequence, developing an algebraic generalization that models the growth of a quadratic function and verifying the inverse relationship of the quadratic relationship. *See the task Aussie Fir Tree for full details.*

LEARNING PLAN & ACTIVITIES:

Students begin the unit by doing a close reading of a piece of informational text *(Refer to resource page for possible texts)* describing mathematical patterns that exist in the world around us. A *Text Rendering or Final Word Protocol* will allow all students to identify some of the "Big Ideas" and access/construct meaning from the text by engaging in a structured discussion with peers. Particular attention will be placed on both *Tier II* and *Tier III* words.

Modified KWLW is an instructional activity for supporting students in developing a framework and actively engaging students in constructing meaning of a topic. The process can be framed by asking the following questions:

1. *What do we think we know?*
2. *Were we correct in our thinking?*
3. *What changed in our thinking?*
4. *What did we learn?*
5. *Write an explanation about what you learned.*

Staircases is the long task that should be embedded between the Initial Assessment (Square Patterns) and the Formative Assessment Lesson Table Tiles. This long task provides a series of differentiated instructional activities that ask students to build various staircases and to determine explicit expressions or rules. Blocks or Unifix Cubes should be available to further differentiate the process of learning. *See the task Staircases for full details.*

Think/Write/Pair/Share is a high leverage strategy that respects individual time to process and organize ideas before engaging in peer-to-peer discussions. This process can be used throughout the unit as a vehicle for students to self reflect, construct new meaning by building on the ideas of others, and strengthen their arguments.

"Stop n Jots" and Journal Entries for Reflection: Using a prompt such as, *"How has my thinking changed as a result of what I have discussed with my peers?" or "How can I improve my argument or explanation using evidence and content vocabulary?"* can provide valuable opportunities for students to tweak their own solutions, during class or for homework, and subsequently, deepen their understanding of content.

Input/Output Tables and "Guess My Rule" Games can help struggling students' build their confidence by providing additional opportunities for students to identify and explain various patterns.

Purposeful Questioning and Feedback are instructional supports that can help refocus students' attention on specific aspects of their work. The table below provides some suggestions based on some common difficulties. Although these error patterns/questions relate to the **Table Tiles Formative Assessment Lesson**, they can be easily modified to address similar misconceptions that are revealed from any other problems or tasks used:

Common Issues	Suggested Questions and Prompts
Student makes unintended assumptions For example: The student has calculated the number of whole tiles required to cover the tabletop, assuming she can split tiles to make quarters and halves as needed. Or: The student uses only quarter tiles to cover the tabletop.	§ *Imagine you can buy tiles that are ready cut. You don't need to cut them up. How many of each type do you need?*
Student makes inaccurate drawing For example: The student divides the whole tabletop into "units" of a whole tile surrounded by four quarter tiles. Or: The student draws freehand with a different	§ *How would you describe how to draw a 30 cm by 30 cm tabletop?*

number of half tiles along each side.	
Student assumes proportionality For example: For Q1 the student writes "10 whole tiles, 8 half tiles, 8 quarter tiles." The student believes a tabletop with sides twice as long will need twice as many tiles of each type.	§ *Read the rubric. Where does Maria use quarter tiles? Half tiles?* § *What happens to tiles in the middle of the diagram if you extend the size?*
Unsystematic work For example: The student draws seemingly unconnected examples, such as 10 cm by 10 cm or 40 cm by 40 cm. Or: The student omits some diagrams, drawing tabletops that are 20 cm by 20 cm, and 40 cm by 40 cm, but not 30 cm by 30 cm.	§ *Which example will you draw next? Why?* § *What do you notice about the difference between the number of whole tiles in one tabletop and the next?* § *The sizes of square tabletops are all multiples of 10 cm. Do your diagrams show this?*
Student does not generalize For example: The student does not seem to know how to proceed with finding the quadratic expression. Or: The student identifies patterns in the numbers of different types of tiles but does not extend to the general case.	§ *Can you describe a visual pattern in the number of whole tiles in consecutive diagrams?* § *How could I find out the number of tiles needed for a larger tabletop, without having to continue the pattern?*
Student does not use algebra For example: The student shows awareness of how the number of whole tiles increases with dimensions, but links this to a specific example rather than identifying variables and forming an equation.	§ *How can you write your answer using mathematical language?*

RESOURCES:

Websites and Web-tools used

- Ø http://www.nsrfharmony.org/resources.html
- Ø http://www.khanacademy.org/
- Ø http://www.mathwire.com/archives/algebra.html

Materials Used

- Ø "Guess My Rule" Activities and Input/Output Tables
- Ø Square Patterns worksheet
- Ø Table Tile worksheet and Grid Paper
- Ø Aussie Fir Tree Performance Task
- Ø Text Rendering or Final Word Protocols (See National School Reform Faculty Website Above)
- Ø Looking at Student Work Protocols (See National School Reform Faculty Website Above)

NYC Department of Education
Dennis M. Walcott, Chancellor

Unit Outline –High School Math

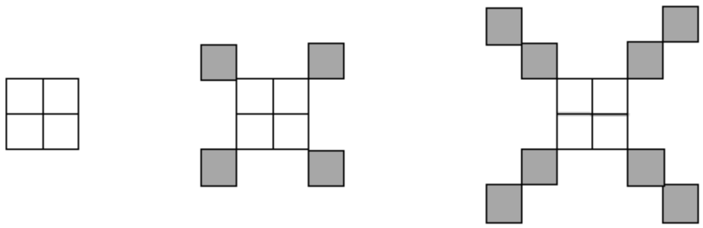
Texts Used(fiction, non-fiction, on-line, media, etc...)

- Ø The Number Devil: A Mathematical Adventure by Hans Magnus Enzensberger
- Ø Life by the Numbers (Chapter 3: “Patterns of Nature”) by Keith Devling
- Ø Innumeracy: Mathematical Illiteracy and Its Consequences (Chapter 1: “Examples and Principles) by John Allen Paulos

Square Patterns

This problem gives you the chance to:
- work with a sequence of tile patterns
- write and use a formula

Mary has some white and gray square tiles.
She uses them to make a series of patterns like these:



Pattern #1   Pattern #2   Pattern #3

1. How many gray tiles does Mary need to make the next pattern?

____

2. What is the total number of tiles she needs to make pattern number 6?

   Explain how you figured it out. _______________

____

____

____

3. Mary uses 48 tiles in all to make one of the patterns.

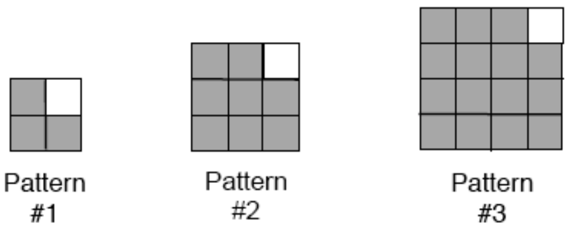
   What is the number of the pattern she makes? _______________

   Show your work.

4. Write a formula for finding the total number of tiles Mary needs to make pattern # n.

____

Mary now uses gray and white square tiles to make a different pattern.



5. How many gray tiles will there be in pattern # 10? ____________

Explain how you figured it out.

____

____

____

6. Write an algebraic formula linking the pattern number, P, with the number of gray tiles, T.

____

Mathematics Assessment Project

Formative Assessment Lesson Materials

Generalizing Patterns:
Table Tiles

MARS Shell Center
University of Nottingham & UC Berkeley
Beta Version

If you encounter errors or other issues in this version, please send details to the MAP team c/o map.feedback@mathshell.org.

Generalizing Patterns: *Table Tiles*

Mathematical goals

This lesson unit is intended to help you assess how well students are able to identify linear and quadratic relationships in a realistic context: the number of tiles of different types that are needed for a range of square tabletops. In particular, this unit aims to identify and help students who have difficulties with:

- Choosing an appropriate, systematic way to collect and organize data.
- Examining the data and looking for patterns; finding invariance and covariance in the numbers of different types of tile.
- Generalizing using numerical, geometrical or algebraic structure.
- Describing and explaining findings clearly and effectively.

Common Core State Standards

This lesson involves a range of *mathematical practices* from the standards, with emphasis on:

7. Look for and make use of structure.
8. Look for and make use of repeated reasoning.

This lesson asks students to select and apply mathematical content from across the grades, including the *content standards:*

F-BF: Build a function that models a relationship between two quantities.

Introduction

The unit is structured in the following way:

- Before the lesson, students attempt the task individually. You then review their work and formulate questions for students to answer in order for them to improve their work.
- At the start of the lesson, students work individually to answer your questions.
- Next, they work collaboratively, in small groups, to produce a better collective solution than those they produced individually. Throughout their work, they justify and explain their decisions to peers.
- In the same small groups, students critique examples of other students' work.
- In a whole-class discussion, students explain and compare the alternative approaches they have seen and used.
- Finally, students work alone again to improve their individual solutions.

Materials required

- Each individual student will need two copies of the worksheet *Table Tiles* and two copies of the *Grid Paper*.
- Each small group of students will need a copy of the *Grid Paper* and a copy of *Sample Responses to Discuss*.
- There are some projectable resources to help you with the whole-class discussions.

Time needed:

Approximately fifteen minutes before the lesson, a one-hour lesson, and ten minutes in a follow-up lesson (or for homework). All timings are approximate. Exact timings will depend on the needs of the class.

Before the lesson

Assessment task: *Table Tiles* (15 minutes)

Have the students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.

Give each student a copy of *Table Tiles* and a copy of the grid paper. Introduce the task briefly and help the class to understand the problem and its context.

> *Spend fifteen minutes on your own, answering these questions.*
>
> *Show your work on the worksheet and the grid paper.*
>
> *Don't worry if you can't do everything. There will be a lesson on this material [tomorrow] that will help you improve your work. Your goal is to be able to answer these questions with confidence by the end of that lesson.*

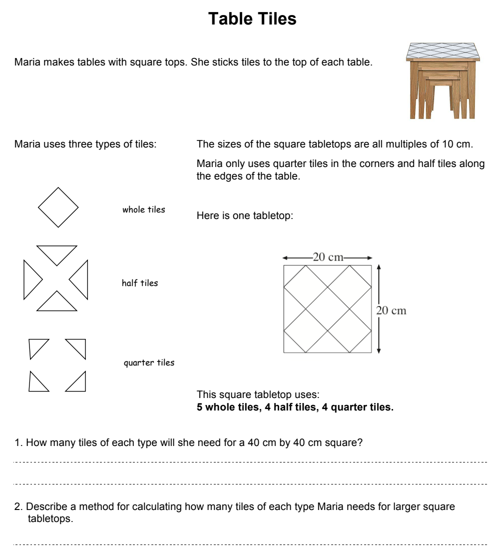


Table Tiles

Maria makes tables with square tops. She sticks tiles to the top of each table.

Maria uses three types of tiles:

whole tiles

half tiles

quarter tiles

The sizes of the square tabletops are all multiples of 10 cm.

Maria only uses quarter tiles in the corners and half tiles along the edges of the table.

Here is one tabletop:

20 cm

20 cm

This square tabletop uses:
5 whole tiles, 4 half tiles, 4 quarter tiles.

1. How many tiles of each type will she need for a 40 cm by 40 cm square?

2. Describe a method for calculating how many tiles of each type Maria needs for larger square tabletops.

It is important that students answer the questions without assistance, as far as possible.

Students who sit together often produce similar answers, and then, when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual places. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this is counterproductive, as it encourages students to compare scores, and distracts their attention from how they may improve their mathematics.

Instead, help students to make further progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you write your own lists of questions, based on your own students' work, using the ideas below. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

Common issues	Suggested questions and prompts
Student makes unintended assumptions For example: The student has calculated the number of whole tiles required to cover the tabletop, assuming she can split tiles to make quarters and halves as needed. Or: The student uses only quarter tiles to cover the tabletop.	• *Imagine you can buy tiles that are ready cut. You don't need to cut them up. How many of each type do you need?*
Student makes inaccurate drawing For example: The student divides the whole tabletop into "units" of a whole tile surrounded by four quarter tiles. Or: The student draws freehand with a different number of half tiles along each side.	• *How would you describe how to draw a 30 cm by 30 cm tabletop?*
Student assumes proportionality For example: For Q1 the student writes "10 whole tiles, 8 half tiles, 8 quarter tiles." The student believes a tabletop with sides twice as long will need twice as many tiles of each type.	• *Read the rubric. Where does Maria use quarter tiles? Half tiles?* • *What happens to tiles in the middle of the diagram if you extend the size?*
Unsystematic work For example: The student draws seemingly unconnected examples, such as 10 cm by 10 cm or 40 cm by 40 cm. Or: The student omits some diagrams, drawing tabletops that are 20 cm by 20 cm, and 40 cm by 40 cm, but not 30 cm by 30 cm.	• *Which example will you draw next? Why?* • *What do you notice about the difference between the number of whole tiles in one tabletop and the next?* • *The sizes of square tabletops are all multiples of 10 cm. Do your diagrams show this?*
Student does not generalize For example: The student does not seem to know how to proceed with finding the quadratic expression. Or: The student identifies patterns in the numbers of different types of tiles but does not extend to the general case.	• *Can you describe a visual pattern in the number of whole tiles in consecutive diagrams?* • *How could I find out the number of tiles needed for a larger tabletop, without having to continue the pattern?*
Student does not use algebra For example: The student shows awareness of how the number of whole tiles increases with dimensions, but links this to a specific example rather than identifying variables and forming an equation.	• *How can you write your answer using mathematical language?*
Student provides a recursive rule not an explicit formula For example: The student provides a way to calculate the number of tiles in a tabletop the next size up from a given size, rather than a general formula for a tabletop of any size; for example, "next = now + 4", or "add two more whole tiles than you did last time."	• *Would your method be practical if I wanted to calculate the number of tiles in a 300 cm by 300 cm tabletop?*

Common issues	Suggested questions and prompts
Student writes incorrect formula For example: The student writes an incorrect formula such as $4x^2+4x+4$ for the number of whole tiles, either using an incorrect algebraic structure or making a recording mistake.	• *Does your formula give the correct number of whole tiles in tabletops of different sizes?*
Student writes answers without explanation	• *How could you explain how you reached your conclusions so that someone in another class understands?*
Student correctly identifies constant, linear, and quadratic sequences	• *Think of another way of solving the problem. Is this method better or worse than your original one? Explain your answer.* • *Can you extend your solution to include rectangular tabletops that aren't squares?*

Suggested lesson outline

Improve individual solutions to the assessment task (10 minutes)

Return your students' work on the *Table Tiles* problem. Ask students to re-read both the *Table Tiles* problem and their solutions. If you have not added questions to students' work, write a short list of your most common questions on the board. Students can then select a few questions appropriate to their own work and begin answering them.

Recall what we were working on previously. What was the task?

Draw students' attention to the questions you have written.

I have read your solutions and I have some questions about your work.

I would like you to work on your own to answer my questions for ten minutes.

Collaborative small-group work (15 minutes)

Organize the students into small groups of two or three. In trials, teachers found keeping groups small helped more students play an active role. Give each group a new sheet of grid paper.

Students should now work together to produce a joint solution.

Put your solutions aside until later in the lesson. I want you to work in groups now.

Your task is to work together to produce a solution that is better than your individual solutions.

You have two tasks during small-group work, to note different student approaches to the task, and to support student problem solving.

Note different student approaches to the task

Notice how students work on finding the quadratic function for the number of whole tiles. Notice also whether and when students introduce algebra. If they do use algebra, note the different formulations of the functions they produce, including incorrect versions, for use in whole-class discussion. You can use this information to focus the whole-class plenary discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions to help students clarify their thinking. If several students in the class are struggling with the same issue, you could write a relevant question on the board. You might also ask a student who has performed well on one part of the task to help a student struggling with that part of the task.

The following questions and prompts have been found most helpful in trials:

What information have you been given?

What do you need to find out?

What changes between these diagrams? What stays the same?

What if I wanted to know the thousandth example?

How will you write down your pattern?

Why do you think your conjecture might be true?

In trials, teachers expressed surprise at the degree of difficulty some students experienced in drawing the tabletops. If this issue arises in your class, help the student to focus his or her attention on different parts of the tabletop, how they align with the grid, and then get them to draw the whole diagram from those pieces.

Where are the half tiles? Whole tiles?

How do the whole and half tiles fit together?

You may find that some students do not work systematically when drawing tabletops and organizing their data.

What sizes of tabletop might Maria make? Which ones is it useful for you to draw? Why?

What can you do to organize your data?

If students have found formulas, focus their attention on improving explanations, exploring alternative methods, and showing the equivalence of different equations.

How can you be sure your explanation works in all cases?

Ask another group if your argument makes sense.

Which is the formula you prefer? Why?

Show me that these two expressions are equivalent.

Students may justify their formulas by drawing another example to see if the generalization fits a new case, reasoning inductively. Some stronger explanations are shown in the *Sample Responses to Discuss*.

Collaborative analysis of *Sample Responses to Discuss* (15 minutes)

Give each small group of students a copy of the *Sample Responses to Discuss*. These are three of the common problem solving approaches taken by students in trials. Display the following questions on the board or OHP using the provided sheet: *Analyzing Student responses to discuss*.

Describe the problem solving approach the student used.

You might, for example:

- *Describe the way the student has colored the pattern of tiles.*
- *Describe what the student did to calculate a sequence of numbers.*

Explain what the student could do to complete his or her solution.

This analysis task will give students an opportunity to evaluate a variety of alternative approaches to the task, without providing a complete solution strategy.

During small-group work, support student thinking as before. Also, check to see which of the explanations students find more difficult to understand. Identify one or two of these approaches to discuss in the plenary discussion. Note similarities and differences between the sample approaches and those the students took in small-group work.

Plenary whole-class discussion comparing different approaches (20 minutes)

Organize a whole-class discussion to consider different approaches to the task. The intention is for you to focus on getting students to understand the methods of working out the answers, rather than either numerical or algebraic solutions. Focus your discussion on parts of the two small-group tasks students found difficult. You may find it helpful to display a copy of the *Grid Paper* or the projector resource *Tabletops.*

Let's stop and talk about different approaches.

Ask the students to compare the different solution methods.

Which approach did you like best? Why?

Which approach did you find it most difficult to understand?

Sami, your group used that method. Can you explain that for us?

Which method would work best for the thousandth tabletop?

Below, we have given details of some discussions that emerged in trial lessons.

Some students found the work on quadratic expressions very difficult. If your students have this problem, you might focus on Gianna's method from the *Sample Responses to Discuss*.

Describe Gianna's pattern in the whole tiles in the 30 cm by 30 cm tabletop.

How would you describe her pattern in the next size tabletop?

Using Gianna's pattern, how many whole tiles would there be in any tabletop?

If students have found different algebraic formulations for the number of half and whole tiles, it might help to write a variety of their expressions on the board. Ask students to link different variables and manipulate algebraic expressions to identify errors and show equivalences:

Which of these formulas would you use to find the number of half tiles?

Which are quadratic?

Are there any formulas that are equivalent?

Review individual solutions to the assessment task (10 minutes)

If you are running out of time, you could schedule this activity for the next lesson or for homework.

Make sure students have their original individual work on the *Table Tiles* task to hand. Give them a fresh, blank copy of the *Table Tiles* task sheet and of the *Grid Paper*.

Read through your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If a student is satisfied with his or her solution, ask the student to try a different approach to the problem and to compare the approach already used.

Solutions

Size of tabletop (cm)	10 × 10	20 × 20	30 × 30	40 × 40	50 × 50	60 × 60
Number of quarter tiles	4	4	4	4	4	4
Number of half tiles	0	4	8	12	16	20
Number of whole tiles	1	5	13	25	41	61

For a tabletop of side length x, $n = \frac{x}{10}$ is the number of tile diagonal widths in the side length.

The number of quarter tiles is always 4. The number of half tiles is $4(n-1) = 4\left(\frac{x}{10} - 1\right)$.

The number of whole tiles is $n^2 + (n-1)^2 = 2n^2 - 2n + 1 = 2\left(\frac{x}{10}\right)^2 - 2\left(\frac{x}{10} - 1\right)^2 = \left(\frac{x}{10}\right)^2 + \left(\frac{x}{10} - 1\right)^2$.

Analysis of *Student Responses to Discuss*

Leon's method

Leon drew three diagrams showing tabletops with systematically increasing side lengths. He colored the diagrams to pick out rows of whole squares parallel to the diagonal of the tabletop.

Width of tabletop	**10 cm**	**20 cm**	**30 cm**
Number of whole tiles	1	1 + 3 + 1	1 + 3 + 5 + 3 + 1

Leon wrote the sum of the numbers to show the total number of whole tiles. He did this in an organized way. He predicted the number of whole tiles in the next diagram accurately.

Leon could check the next diagram to see if his conjecture is correct, but this would not be a proof. He could use his method to predict the number of whole tiles in the next diagram for any diagram he has drawn, but that method would not enable him to calculate the number of whole tiles for tables of any size. To do that, he might attempt to articulate the relationship between the number of whole tiles across the diagonal and the number of odd numbers to sum (assuming that students do not recognize that $1 + 3 + 5 + \ldots + 2n - 1 = n^2$).

Gianna's method

Gianna shaded alternate horizontal rows of squares. In the first two diagrams she wrote numbers in the whole squares. These record the number of whole squares in each horizontal row of that tabletop.

For the diagram of the 30 cm tabletop this gives:

In the first row:	3		3		3
In the second row:		2		2	
In the third row:	3		3		3
In the fourth row:		2		2	
In the fifth row:	3		3		3

Gianna picked out the pattern as 3 rows of 3 whole tiles, and 2 rows of 2 whole tiles, to find the total number of whole tiles is $3 \times 3 + 2 \times 2$.

Gianna could then generalize to show that in the n^{th} tabletop there would be n rows of n whole tiles, and n-1 rows of n-1 whole tiles, in total $n^2+(n-1)^2$ whole tiles.

Ava's method

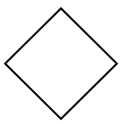
Ava drew one diagram showing how the length of a 40 cm tableside is made of two 5 cm edges of quarter tiles and three 10cm half tiles, and another similar diagram for the 50 cm table. Ava systematically organized data in a table, although it is not clear where the data came from. She found differences and second differences between numbers of tiles in tables of increasing side length. She did not show a way of calculating the number of tiles of various types given an arbitrary table size, and she did not use algebra.

Ava could next use her table of data to derive formulas for the number of tiles in any table. She might do this by using the first and second differences with the standard quadratic sequence algorithm.

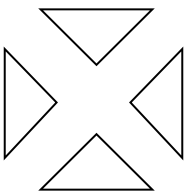
Table Tiles

Maria makes tables with square tops. She sticks tiles to the top of each table.

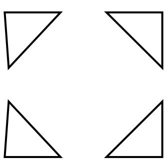
Maria uses three types of tiles:



whole tiles



half tiles

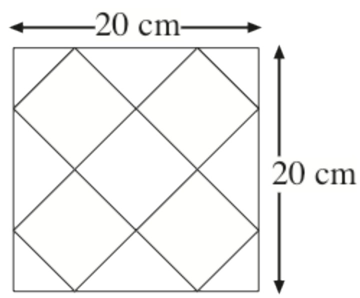


quarter tiles

The sizes of the square tabletops are all multiples of 10 cm.

Maria only uses quarter tiles in the corners and half tiles along the edges of the table.

Here is one tabletop:

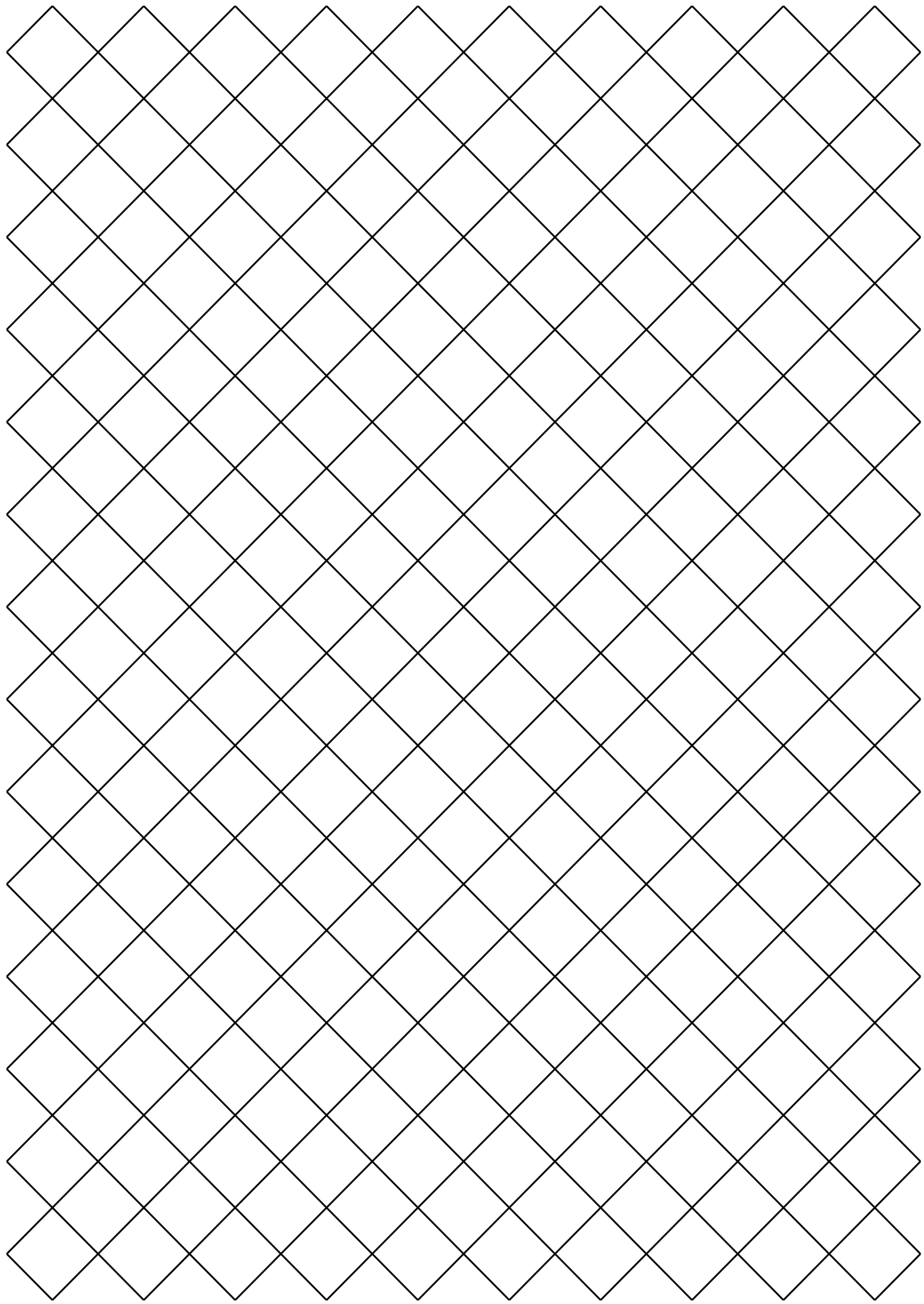


This square tabletop uses:
5 whole tiles, 4 half tiles, 4 quarter tiles.

1. How many tiles of each type will she need for a 40 cm by 40 cm square?

2. Describe a method for calculating how many tiles of each type Maria needs for larger square tabletops.

Grid Paper



Sample Responses to Discuss

Here is some work on *Table Tiles* from three students in another class: Leon, Ava and Gianna.

For each piece of work:

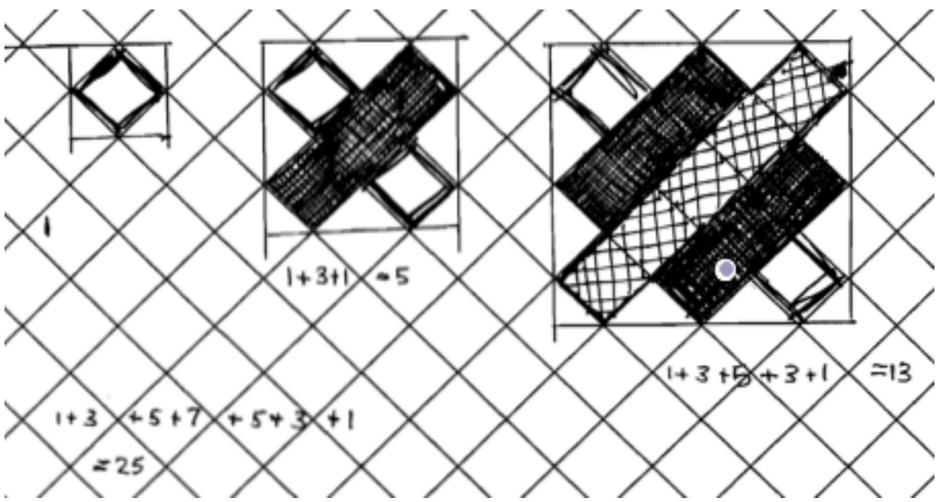
1. Describe the problem solving approach the student used.

   For example, you might:

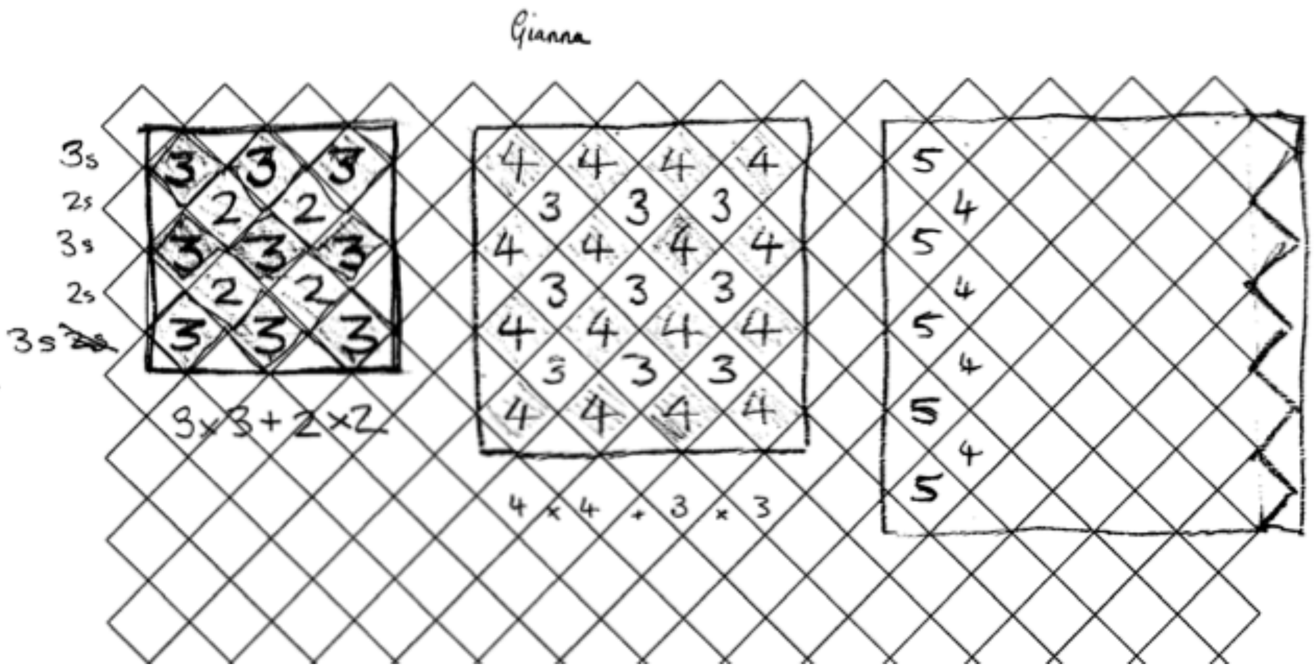
   - Describe the way the student has colored the pattern of tiles.
   - Describe what the student did to calculate a sequence of numbers.

2. Explain what the student needs to do to complete his or her solution.

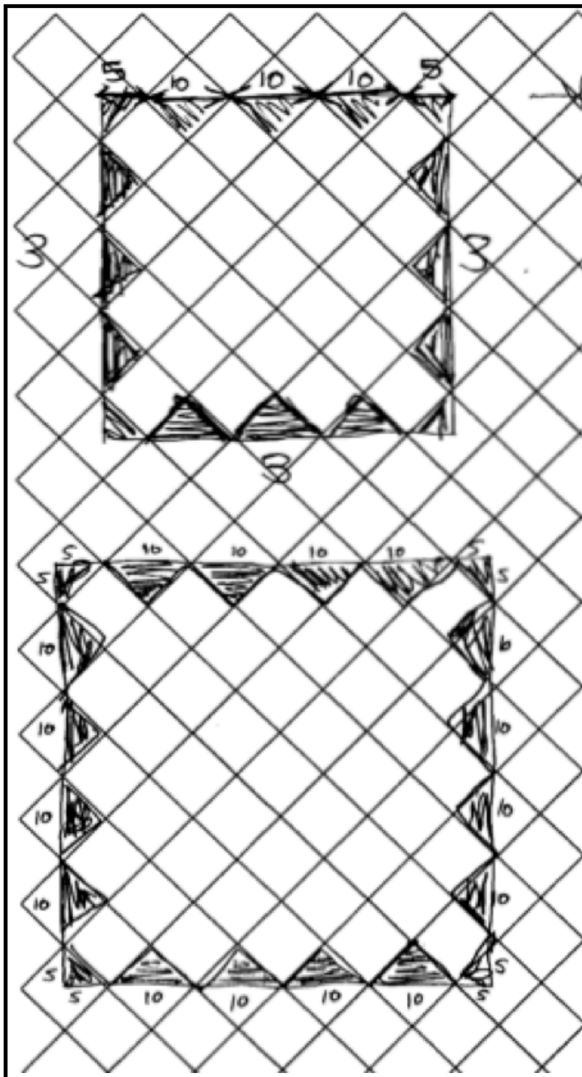
Leon's method



Gianna's method



Ava’s method

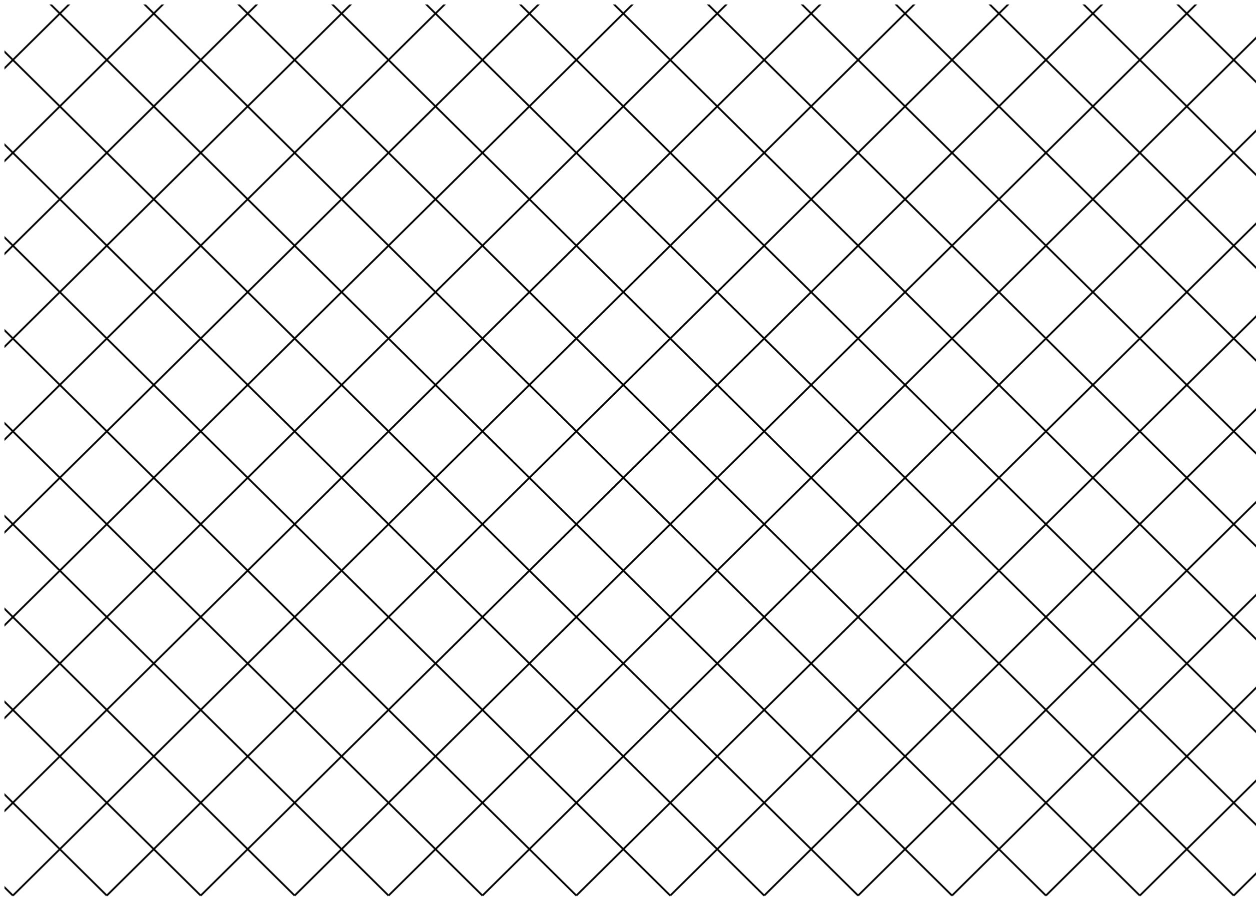


Side length	10cm	20cm	30cm	40cm	50cm
Quarters	4	4	4	4	4
Halfs	0	4	8	12	16
Wholes	1	5	13	25	41

4 8 12 16

+4 +4 +4

Grid Paper



Student Responses to Discuss

1. Describe the problem solving approach the student used.

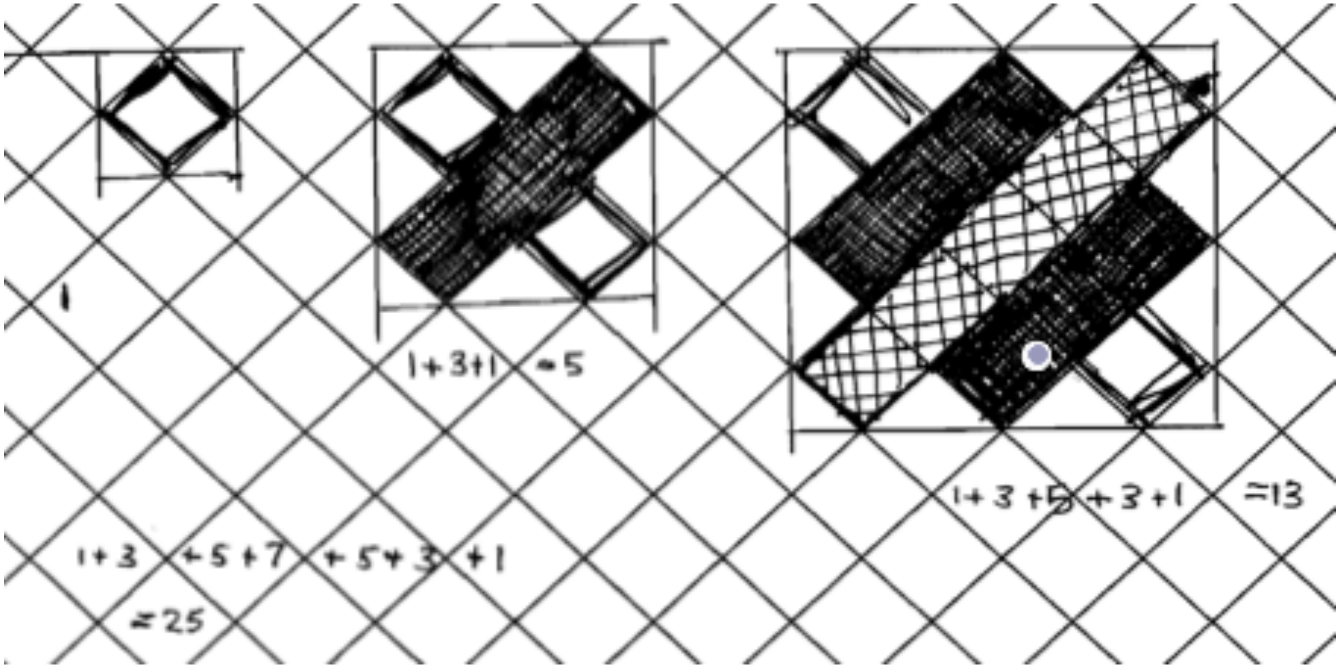
   You might, for example:

   Describe the way the student has colored the pattern of tiles.

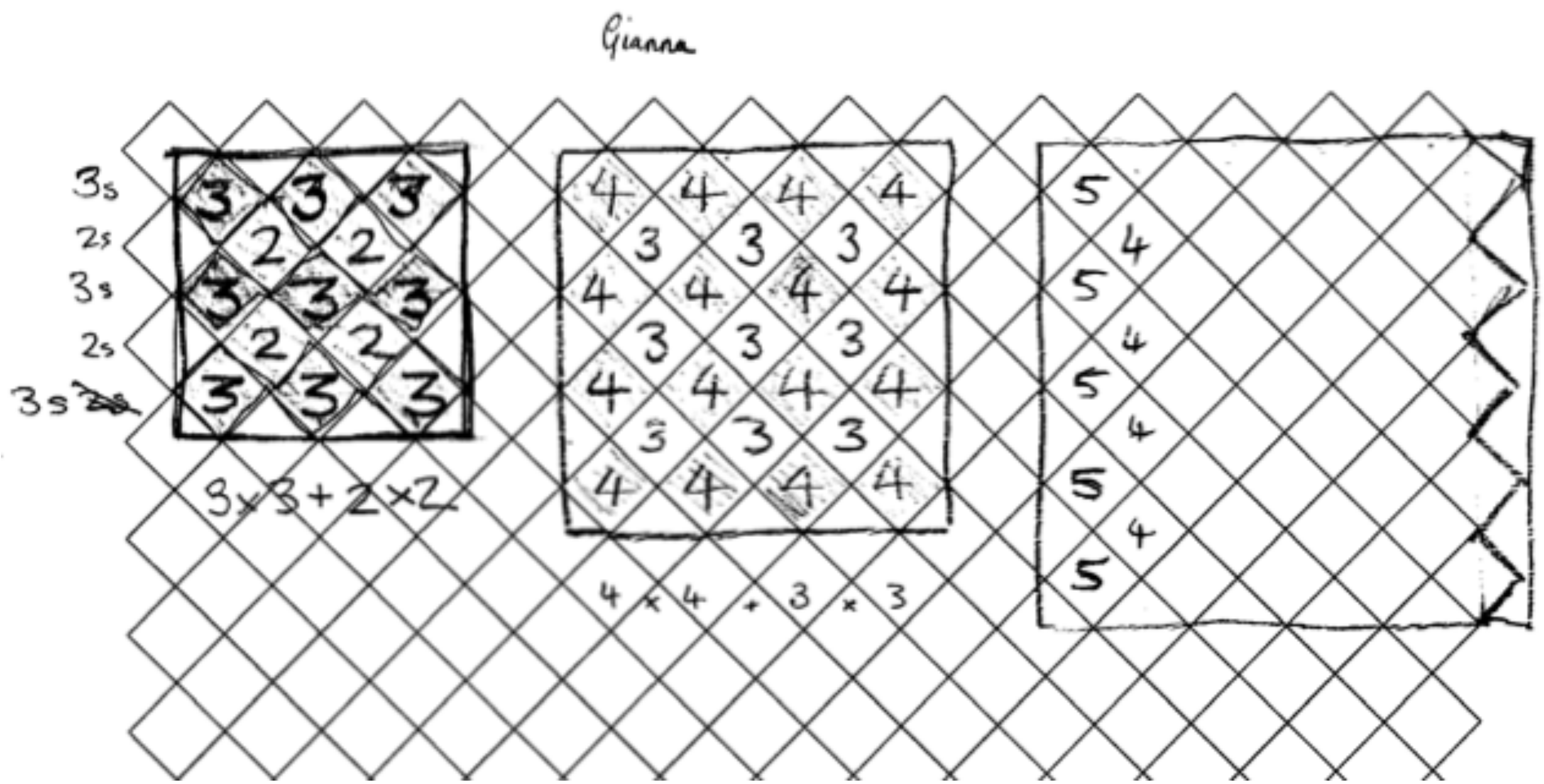
   Describe what the student did to calculate a sequence of numbers.

2. Explain what the student needs to do to complete his or her solution.

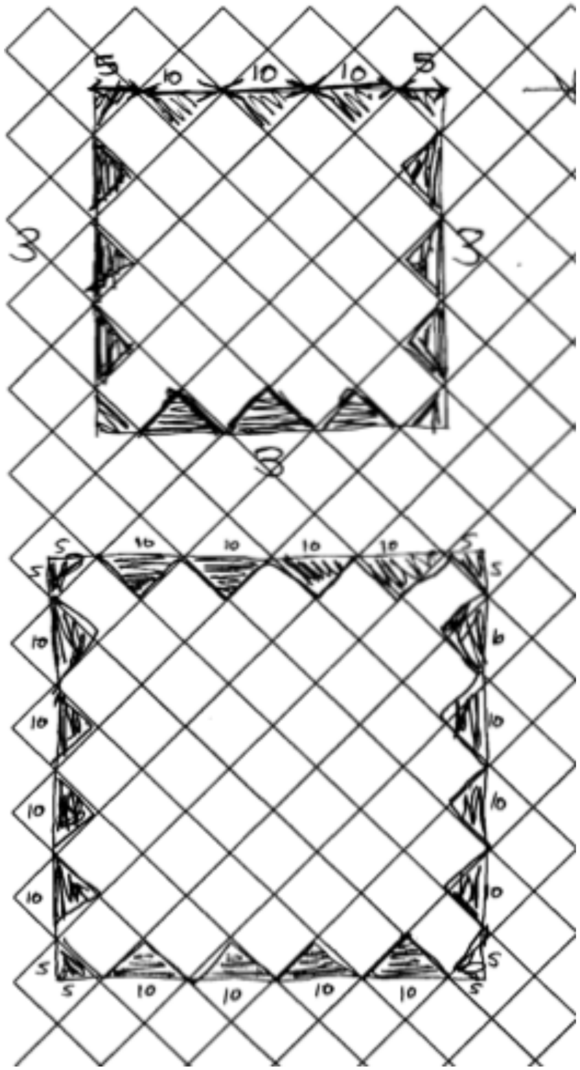
Leon’s method



Gianna's method



Ava's method

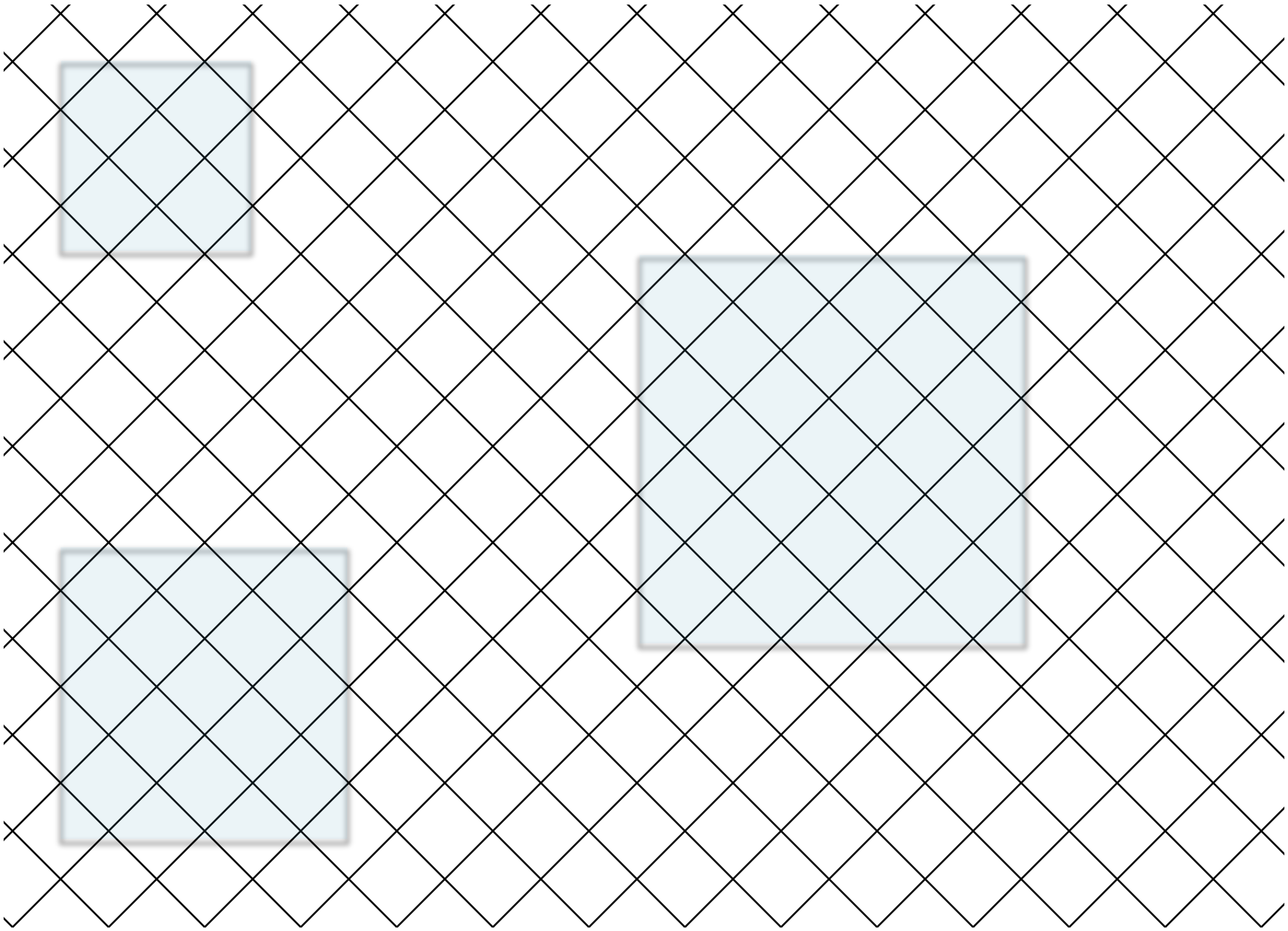


Side length	10cm	20cm	30cm	40cm	50cm
Quarters	4	4	4	4	4
Halfs	0	4	8	12	16
Wholes	1	5	13	25	41

4 8 12 16

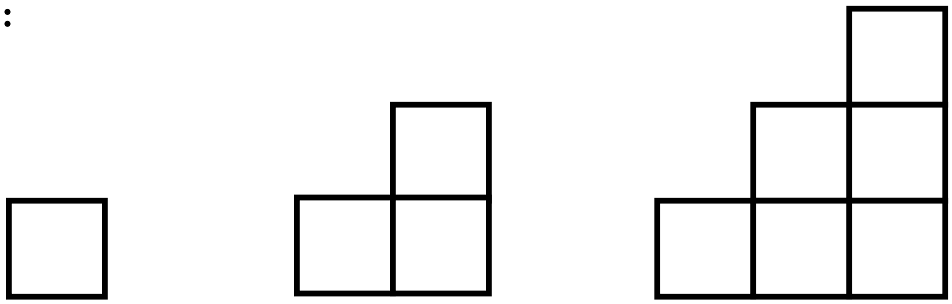
+4 +4 +4

Tabletops



Staircases

Level C:



Staircase 1
1 block in all
First step 1 block

Staircase 2
3 blocks in all
Second step has 2 blocks

Staircase 3
6 blocks in all
Third step has 3 blocks

This is a staircase that goes up steps.

How many blocks are needed for the fourth staircase?

How many blocks are needed to make just the one hundredth step? Explain how you know.

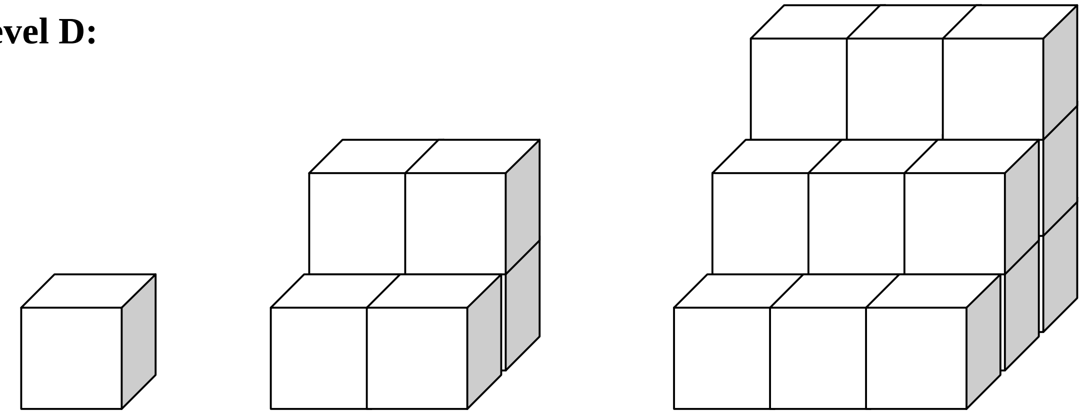
Write a rule to find the number of blocks needed for the nth step? Explain your rule.

Write a rule to find the total number of blocks needed to make a staircase with n number of steps. Explain your rule.

If you had 300 blocks to build the largest staircase possible, how many steps are in the staircase. Explain your thinking.

Staircases

Level D:



Staircase 1 **Staircase 2** **Staircase 3**

This set of staircases grows at a different rate.

How many blocks, in all, are needed to make a staircase with five steps?

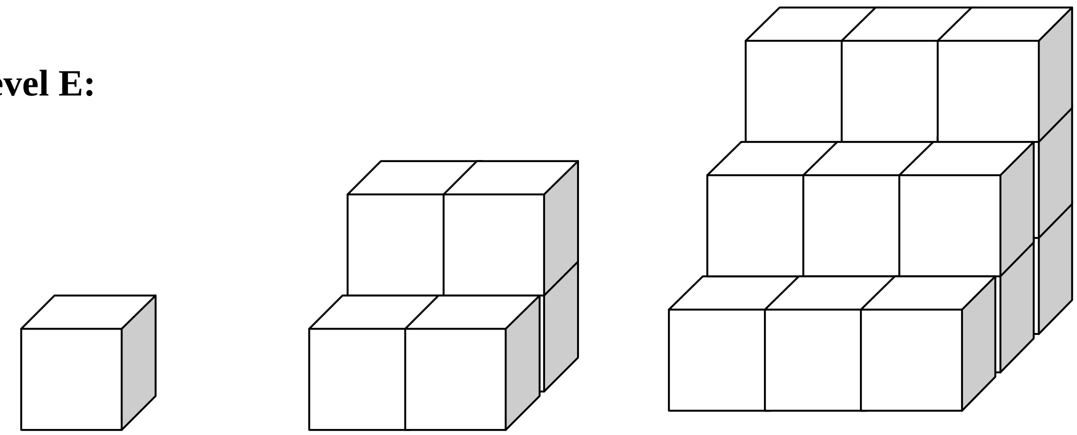
How many blocks make up the top step of a staircase with n steps?

How many blocks make up the first level (the base) of a staircase with n steps?

Given a staircase with 30 steps explain a process you might follow to determine the number of blocks necessary to build the staircase. Explain your answers.

Staircases

Level E:



Staircase 1 **Staircase 2** **Staircase 3**

Using the pattern shown above, find a general (closed) formula to find the number of blocks needed to build a staircase with n stairs.

Justify why your formula works.

Explain and justify which staircases will require an odd number of blocks to build them.